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B. Sc. (Hons) Part-II, Paper- IV

Differential Equation (First order First degree)

Homogeneous Equations:- A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

Where $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of x, y and of the same degree is said to be homogeneous.

Solving Rule:- Every homogeneous equation of the above type can be solved by putting $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

After making the substitution for y and $\frac{dy}{dx}$, the given differential equation can be reduced to the form in which the variables are separable. Then after integration we replace y by $\frac{v}{x}$ and the required result follows.

Ex. ① Solve $x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$

Soln. From the given eqn. we have

$$\frac{dy}{dx} = \frac{y + x\sqrt{x^2+y^2}}{x}, \text{ Put } y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x} = \cancel{v} + x\sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = x\sqrt{1+v^2} \Rightarrow dx = \frac{dv}{\sqrt{1+v^2}} \Rightarrow \int dx = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow x + c = \sinh^{-1} v \Rightarrow v = \sinh(x+c)$$

$$\Rightarrow \frac{y}{x} = \sinh(x+c) \Rightarrow y = x\sinh(x+c)$$

which is reqd. soln.

Ex ② Solve $(x^2-y^2)\frac{dy}{dx} = 2xy$

Soln. Given $(x^2-y^2)\frac{dy}{dx} = 2xy$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \qquad \text{——— } (1)$$

Put $y = vx$, So that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence the eqn. (1) becomes.

$$v + x\frac{dv}{dx} = \frac{2x \cdot vx}{x^2 - v^2x^2} = \frac{2vx^2}{x^2(1-v^2)} = \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2} = \frac{v+v^3}{1-v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-v^2}{v+v^3}\,dv = \frac{1-v^2}{v(1+v^2)}\,dv$$

Now, integrating, we get

$$\int \frac{dx}{x} = \int \frac{1-v^2}{v(1+v^2)}\,dv = \int \left(\frac{1}{v} - \frac{2v}{1+v^2}\right) dv$$

$$\Rightarrow \log x = \log v - \log(1+v^2) + K$$

$$= \log\frac{y}{x} - \log\left(1 + \frac{y^2}{x^2}\right) + K$$

$$= (\log y - \log x) - \{\log(x^2+y^2) - \log x^2\} + K$$

$$= \log y - \log x - \log(x^2+y^2) + 2\log x + K$$

$$= \log y + \log x - \log(x^2+y^2) + K$$

$$\Rightarrow \log(x^2+y^2) = \log y + \log x - \log x + K$$

$$\Rightarrow \log(x^2+y^2) = \log y + K = \log y + \log c \quad [K = \log c]$$

$$\Rightarrow \log(x^2+y^2) = \log cy$$

$$\Rightarrow x^2 + y^2 = cy$$

which is the required soln.